

Cosmic ray small-scale anisotropies in quasi-linear theory

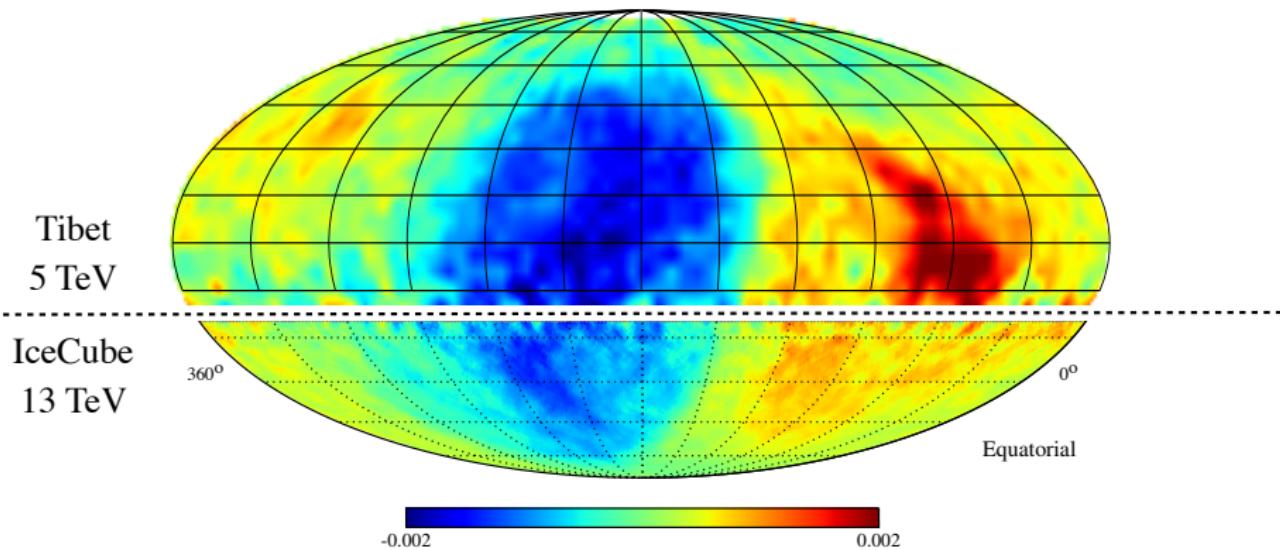
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with Markus Ahlers

ICRC 2019
Madison, WI, 26 July 2019



Cosmic ray anisotropies

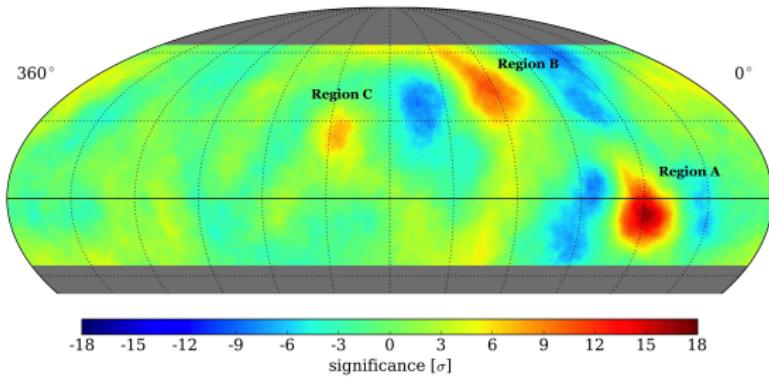
$$I(\mathbf{n}) \equiv \frac{\phi(\mathbf{n})}{\phi^{\text{iso}}} \equiv 1 + \delta I(\mathbf{n})$$



Amenomori *et al.*, ApJ 711 (2010) 119, Saito *et al.*, Proc. 32nd ICRC 1 (2011) 62

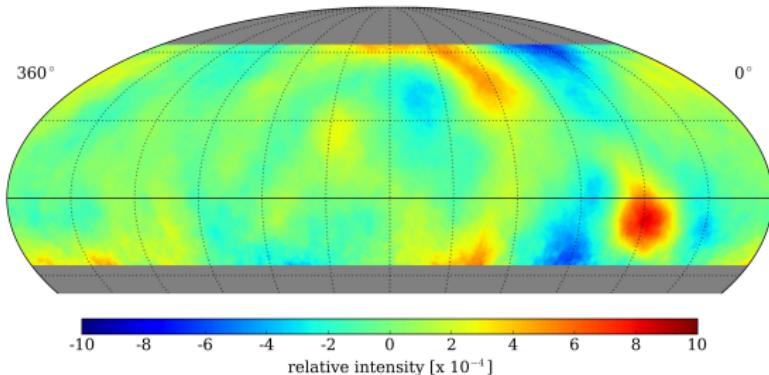
Aartsen *et al.*, ApJ 826 (2016) 220

Small-scale anisotropies



- Subtract off dipole and quadrupole
- Smooth with 10° disk
- Small-scale features

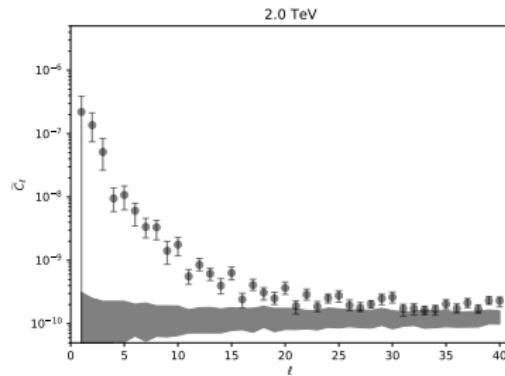
Abeysekara et al., ApJ 796 (2014) 108



Angular power spectrum

HAWC

Abeysekara *et al.*, ApJ. 865 (2018) 57;
also Abeysekara *et al.*, ApJ 796 (2014) 108

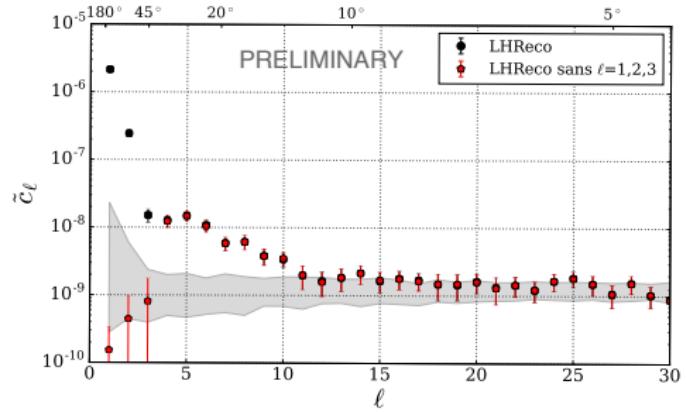


IceCube

Aartsen *et al.*, ApJ 826 (2016) 220

IceCube+HAWC

Daz-Velez *et al.*, Proc. 35th ICRC (2017) 28



Score sheet

Properties

- Large-scale anisotropy of the order $10^{-3} \dots 10^{-4}$ at TeV ... PeV energies
- Small-scale anisotropy of similar size
- Directional pattern also changes with energy
- No time-dependence

Interpretations

Conventional quasi-linear theory only predicts dipole!

- **Magnetic lenses**

Salvati & Sacco, A&A 485 (2008) 527; Drury & Aharonian, Astropart. Phys. 29 (2008) 420; Battaner *et al.*, A&A 527 (2011) 5; Harding *et al.*, ApJ 822 (2016) 102

- **Non-uniform pitch-angle diffusion**

Malkov *et al.*, ApJ 721 (2010) 750; Giacinti & Kirk, ApJ 835 (2017) 258

- **Heliospheric effects**

Lazarian & Desiati, ApJ 722 (2010) 188; Desiati & Lazarian, ApJ 762 (2013) 44; Drury, Proc. 33rd ICRC (2013); Zhang *et al.*, ApJ 790 (2014) 5

- **Small-scale turbulence**

Giacinti & Sigl, PRL 109 (2012) 071101; Ahlers, PRL 117 (2016) 151103; Ahlers & Mertsch, ApJL 815 (2015) L2; Pohl & Rettig, Proc. 36th ICRC (2015) 451; López-Barquero *et al.*, ApJ 830 (2016) 19; López-Barquero *et al.*, ApJ 842 (2017) 54

- **Exotics** Kotera *et al.*, Phys. Lett. B725 (2013) 196; Harding, arXiv:1307.6537

Small-scale turbulence and ensemble averaging

- In standard diffusion, compute C_ℓ from $\langle f \rangle$:

$$C_\ell^{\text{std}} = \frac{1}{4\pi} \int d\hat{\mathbf{p}}_1 \int d\hat{\mathbf{p}}_2 P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle$$

- However, in an individual realisation of δB , $\delta f = f - \langle f \rangle \neq 0$

$$\langle C_\ell \rangle = \frac{1}{4\pi} \int d\hat{\mathbf{p}}_1 \int d\hat{\mathbf{p}}_2 P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle$$

- If $f(\hat{\mathbf{p}}_1)$ and $f(\hat{\mathbf{p}}_2)$ are correlated,

$$\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle \geq \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle \quad \Rightarrow \quad \langle C_\ell \rangle \geq C_\ell^{\text{std}}$$

Source of the small scale anisotropies?

Giacinti & Sigl, PRL 109 (2012) 071101

Ahlers, PRL 112 (2014) 021101, Ahlers & Mertsch, ApJL 815 (2015) L2, Pohl & Rettig, Proc. 36th ICRC (2015) 451, López-Barquero *et al.*, ApJ 830 (2016) 19, López-Barquero *et al.* ApJ 842 (2017) 54

Gradient ansatz

- Vlasov equation:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} f + \frac{q\mathbf{v}}{c} \times (\mathbf{B}_0 + \delta\mathbf{B}) \cdot \nabla_{\mathbf{p}} f \\ &\simeq \frac{\partial f}{\partial t} + \underbrace{(c\hat{\mathbf{p}} \cdot \nabla_{\mathbf{r}} + q(\hat{\mathbf{p}} \times \mathbf{B}_0) \cdot \nabla_{\mathbf{p}})}_{\mathcal{L}_0} f + \underbrace{(q(\hat{\mathbf{p}} \times \delta\mathbf{B}) \cdot \nabla_{\mathbf{p}})}_{\delta\mathcal{L}} f = 0\end{aligned}$$

- Gradient ansatz:

$$f(\mathbf{r}, \hat{\mathbf{p}}, t) = f_{\odot}(\hat{\mathbf{p}}, t) + (\mathbf{r}_{\odot} - \mathbf{r}) \cdot \mathbf{G},$$

→ Dipolar source term in the Vlasov equation:

$$\frac{\partial f_{\odot}}{\partial t} + \underbrace{(q(\hat{\mathbf{p}} \times \mathbf{B}_0) \cdot \nabla_{\mathbf{p}})}_{\mathcal{L}'_0} f_{\odot} + \underbrace{(q(\hat{\mathbf{p}} \times \delta\mathbf{B}) \cdot \nabla_{\mathbf{p}})}_{\delta\mathcal{L}} f_{\odot} = c\hat{\mathbf{p}} \cdot \mathbf{G}$$

Mixing matrices

- Define propagator:

$$U_{t,t_0} = \mathcal{T} \exp \left[- \int_{t_0}^t dt' (\mathcal{L}'_0 + \delta\mathcal{L}(t')) \right]$$

- Formal solution of Vlasov equation:

$$f_\odot(\mathbf{p}, t) = U_{t,t_0} f_\odot(\mathbf{p}, t_0) + \int_{t_0}^t dt' U_{t,t'} c \hat{\mathbf{p}} \cdot \mathbf{G}$$

→ Differential equation for $\langle C_\ell \rangle$,

$$\frac{d}{dt} \langle C_\ell \rangle(t) + \left(\lim_{t_0 \rightarrow t} \frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(t, t_0)}{t - t_0} \right) \langle C_{\ell_0} \rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1}$$

mixing $\ell_0 \rightarrow \ell$

sourcing ℓ

where

$$M_{\ell\ell_0}(t, t_0) = \frac{1}{4\pi} \int d\hat{\mathbf{p}}_A \int d\hat{\mathbf{p}}_B P_\ell(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B) \langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle \frac{2\ell_0 + 1}{4\pi} P_{\ell_0}(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B)$$

One particle propagator

“Feynman” rules

- Free propagator:

$$U_{t,t'}^{(0)}$$



- Stochastic field:

$$\delta\mathcal{L}(t)$$



- Correlation:

$$\langle \delta\mathcal{L}(t) U_{t,t'}^{(0)} \delta\mathcal{L}(t') \rangle$$



$$\begin{aligned} \langle U_{t,t_0} \rangle \equiv & \text{---} = \text{---} + \text{---} + \text{---} \\ & + \text{---} + \text{---} + \dots \end{aligned}$$

The equation shows the definition of the one-particle propagator $\langle U_{t,t_0} \rangle$ as a sum of Feynman diagrams. The first term is a simple horizontal line segment. Subsequent terms are represented by horizontal lines with various loops attached, indicating higher-order corrections in perturbation theory.

Double propagator

For $\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle$ we need correlated evolution of two particles:

$$\begin{aligned} \langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle = & \text{---} + \left(\text{---} + \text{---} + \text{---} \right) \\ & + \left(\text{---} + \text{---} + \text{---} + \text{---} \right. \\ & + \text{---} + \text{---} + \text{---} + \text{---} \\ & \left. + \text{---} + \text{---} \right) + \dots \end{aligned}$$

The diagrams show various configurations of two horizontal lines (propagators) with dots representing vertices. Some lines have dashed arcs above or below them, representing loops or external fields. The first row shows three simple diagrams: a single loop, a line with a loop attached to one vertex, and a line with a loop attached to the other vertex. The subsequent rows show more complex configurations involving multiple loops and interactions between the two lines.

Ignoring correlations

- Without “interactions”:

$$\langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle \simeq \text{---} + \text{---} + \text{---} + \text{---}$$

- Mixing matrix diagonal:

$$M_{\ell\ell_0}(t, t_0) \sim \delta_{\ell\ell_0}$$

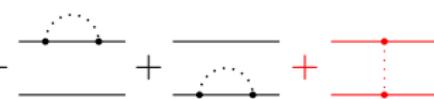
$$\frac{d}{dt} \langle C_\ell \rangle(t) + \left(\lim_{t_0 \rightarrow t} \frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(t, t_0)}{t - t_0} \right) \langle C_{\ell_0} \rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1},$$

- Steady state \rightarrow only dipolar anisotropy:

$$\langle C_\ell \rangle \propto \delta_{\ell 1},$$

With correlations

- With “interactions”

$$\langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle \simeq \text{---} + \text{---} + \text{---} + \text{---}$$


- Mixing matrix **not** diagonal:

$$M_{\ell\ell_0}(t, t_0) \sim \delta_{\ell\ell_0} + \sum_{\ell_A} \kappa_{\ell_A}(t - t_0) \begin{pmatrix} \ell & \ell_A & \ell_0 \\ 0 & 0 & 0 \end{pmatrix}^2 (2\ell_0 + 1)\ell_0(\ell_0 + 1)$$

$$\frac{d}{dt} \langle C_\ell \rangle(t) + \left(\lim_{t_0 \rightarrow t} \frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(t, t_0)}{t - t_0} \right) \langle C_{\ell_0} \rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1},$$

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→ Gradient source term is mixing into higher harmonics!

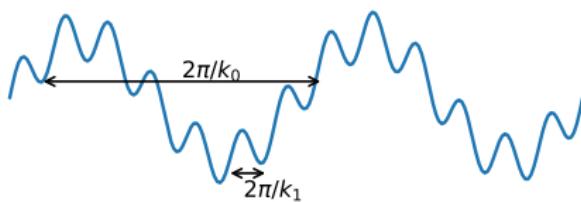
Toy model

- Isotropic turbulence tensor:

$$\langle \delta \tilde{B}_i(\mathbf{k}) \delta \tilde{B}_j^*(\mathbf{k}') \rangle = \frac{g(k)}{k^2} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) \delta(\mathbf{k} - \mathbf{k}')$$

- Band-limited white noise:

$$g(k) = g_0 \quad \text{if} \quad k_0 \leq k < k_1$$

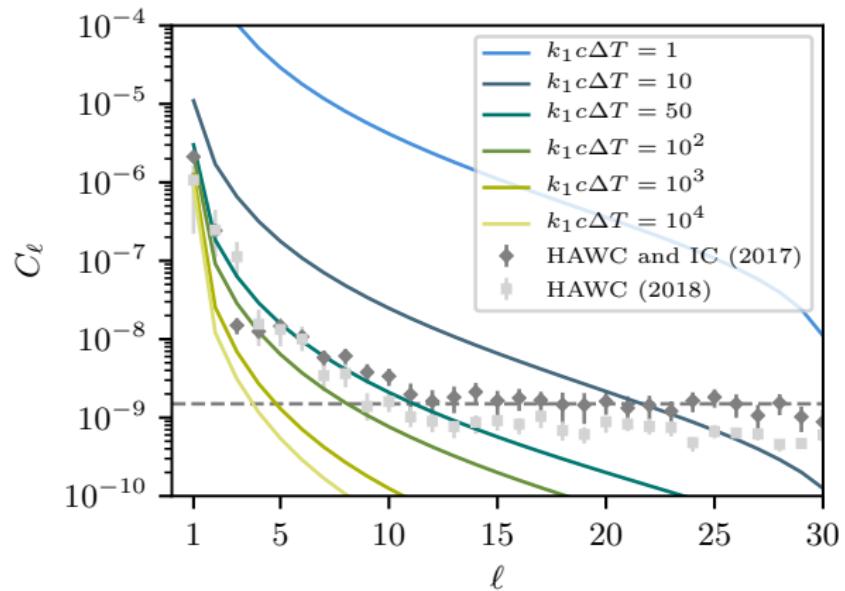


- In order to get local operators

$$\Delta T \equiv (t - t_0) \rightarrow 0 \quad \text{while} \quad k_1 \Delta T = \text{const.}$$

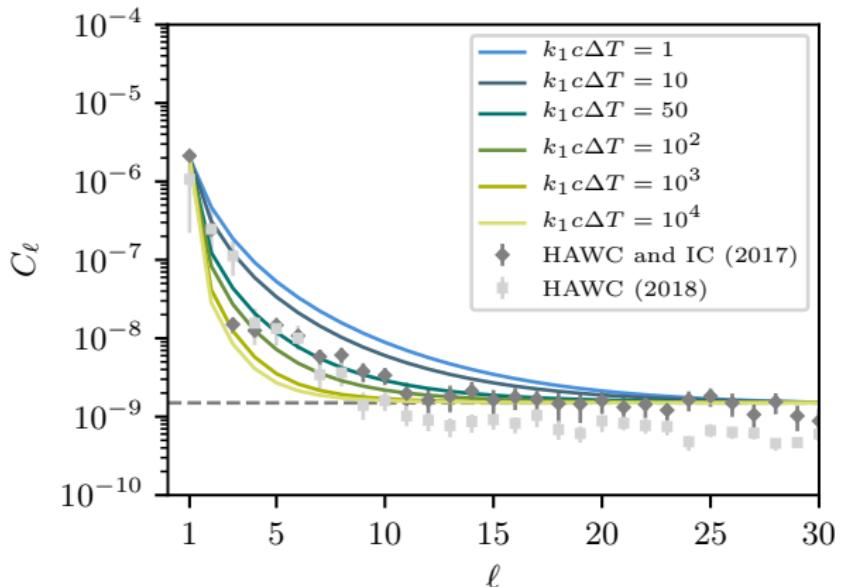
- Require $k_1 \Delta T > 1$ and $k_0 \Delta T \ll 1$

Results



- Fix source term $K|\mathbf{G}|^2$ to $10^{-4}k_0$
- Let $k_1 c \Delta T$ vary
- No shot noise

Results



- Let source term $K|\mathbf{G}|^2$ float
- Numerical simulations point to $k_1 c \Delta \sim 50$
- Add shot noise due to experimental statistics

Discussion

Good agreement with HAWC and combined data

→ Beware of cosmic variance:

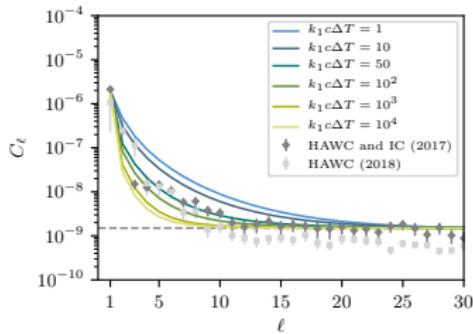
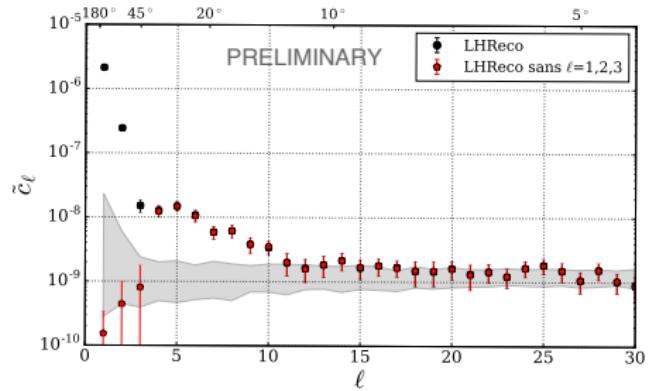
$$\Delta C_\ell = \sqrt{2/(2\ell + 1)} \langle C_\ell \rangle$$

- The anisotropy of the ensemble average might not be perfectly dipolar
Giacinti & Kirk, ApJ 835 (2017) 258
- Need to include regular field B_0
- Test different turbulence correlation tensors

Summary

Observation of anisotropies down to $\lesssim 10^\circ$

Unexpected in conventional diffusion theory!



Small-scale turbulence model

- Correlated propagation of particle pairs
- Differential equation for $\langle C_\ell \rangle$
- Diagrammatic technique

Backup

③ Diagrammatic technique

④ Observations

⑤ Quasi-linear theory

Backup

③ Diagrammatic technique

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Time evolution operator

- Liouville's theorem:

$$\frac{\partial}{\partial t} f + (\mathcal{L}_0 + \delta\mathcal{L}(t)) f(t) = 0$$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle - (H_0 + H_I) |\psi(t)\rangle = 0$$

- Formally solved as

$$f(\mathbf{r}, \mathbf{p}, t) = U_{t, t_0} f(\mathbf{r}, \mathbf{p}, t_0)$$

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle$$

- With free propagator:

$$U_{t, t_0}^{(0)} = \exp \left[- \int_{t_0}^t dt' \mathcal{L}_0(t') \right]$$

$$U^{(0)}(t, t_0) = \exp [-iH_0(t - t_0)/\hbar]$$

- And time evolution operator:

$$U_{t, t_0} = U_{t, t_0}^{(0)} \mathcal{T} \exp \left[- \int_{t_0}^t dt' \underbrace{\left(U_{t', t_0}^{(0)} \right)^{-1} \delta\mathcal{L}(t') U_{t', t_0}^{(0)}}_{\sim \text{interaction picture Hamiltonian}} \right]$$

Mean Green's function

- Perturbative expansion (Dyson series):

$$U_{t,t_0} = U_{t,t_0}^{(0)} + \sum_{n \geq 1} (-1)^n \int_{t_0}^t dt_n \int_{t_0}^{t_n} dt_{n-1} \dots \int_{t_0}^{t_2} dt_1 \\ \times U_{t,t_n}^{(0)} \delta\mathcal{L}(t_n) U_{t_n,t_{n-1}}^{(0)} \delta\mathcal{L}(t_{n-1}) \dots \delta\mathcal{L}(t_1) U_{t_1,t_0}^{(0)}.$$

- But $\delta\mathcal{L}(t)$ is a random variable. So what is $\langle U_{t,t_0} \rangle$?
- Evaluate expectation values in Gaussian approximation:
 $\langle \delta\mathcal{L}(t_n) \delta\mathcal{L}(t_{n-1}) \dots \delta\mathcal{L}(t_1) \rangle \simeq \langle \delta\mathcal{L}(t_n) \delta\mathcal{L}(t_{n-1}) \rangle \dots \langle \delta\mathcal{L}(t_1) \delta\mathcal{L}(t_0) \rangle + \text{permut.}$

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- Fourth order term:

$$\int_{t_0}^{t < t_4 < t_3 < t_2} dt_4 \dots dt_1 U_{t,t_4}^{(0)} \overline{\delta\mathcal{L}(t_4) U_{t_4,t_3}^{(0)} \delta\mathcal{L}(t_3)} U_{t_3,t_2}^{(0)} \overline{\delta\mathcal{L}(t_2) U_{t_2,t_1}^{(0)} \delta\mathcal{L}(t_1)} U_{t_1,t_0}^{(0)} \\ + \int_{t_0}^{t < t_4 < t_3 < t_2} dt_4 \dots dt_1 U_{t,t_4}^{(0)} \overline{\delta\mathcal{L}(t_4) U_{t_4,t_3}^{(0)} \delta\mathcal{L}(t_3)} U_{t_3,t_2}^{(0)} \overline{\delta\mathcal{L}(t_2) U_{t_2,t_1}^{(0)} \delta\mathcal{L}(t_1)} U_{t_1,t_0}^{(0)} \\ + \int_{t_0}^{t < t_4 < t_3 < t_2} dt_4 \dots dt_1 U_{t,t_4}^{(0)} \overline{\delta\mathcal{L}(t_4) U_{t_4,t_3}^{(0)} \delta\mathcal{L}(t_3)} U_{t_3,t_2}^{(0)} \overline{\delta\mathcal{L}(t_2) U_{t_2,t_1}^{(0)} \delta\mathcal{L}(t_1)} U_{t_1,t_0}^{(0)}$$

Resummation and Bourret approximation

- Full series convergent, but partial series can diverge
→ Resummation of connected diagrams into “mass operator”

$$\text{---} = \text{---} + \text{---} + \text{---} + \dots,$$

so summands in $\langle U_{t,t_0} \rangle$ factorise:

$$\text{---} = \text{---} + \text{---} + \text{---} + \dots$$

- Bourret approximation: approximate mass operator with its first term,

$$\text{---} \equiv \text{---} + \text{---} + \text{---} + \dots$$

Diffusion on sphere

- For homogeneous and static turbulence and $\Omega = 0$:

$$\langle U_{t,t_0} \rangle \simeq e^{-\nu(t-t_0)\mathbf{L}^2}$$

- Diffusion equation in $\hat{\mathbf{n}}$:

$$\frac{\partial}{\partial t} f(t, \hat{\mathbf{n}}) - \nu \Delta f(t, \hat{\mathbf{n}}) = 0$$

- Laplacian on sphere (for $|\mathbf{r}| = 1$):

$$\Delta = -\mathbf{L}^2$$

- Solved by:

$$f(t, \hat{\mathbf{n}}) = e^{-\nu(t-t_0)\mathbf{L}^2} f(t_0, \hat{\mathbf{n}}) = \langle U_{t,t_0} \rangle f(t_0, \hat{\mathbf{n}})$$

Bourret propagator describes isotropic pitch-angle scattering

Example: non-interacting diagram

$$\langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle \simeq \text{---} + \text{---} + \text{---} + \text{---}$$

$$\begin{aligned}
 \left(\langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle \right)_{1a} &= \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 U_{t,t_2}^{A(0)} \langle \delta \mathcal{L}_{t_2}^A U_{t_2,t_1}^{A(0)} \delta \mathcal{L}_{t_1}^A \rangle U_{t_1,t_0}^{A(0)} U_{t,t_0}^{B*(0)} \\
 &= - \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \int d^3k \int d^3k' e^{i\mathbf{k} \cdot \mathbf{r}_A(t_2) - i\mathbf{k}' \cdot \mathbf{r}_A(t_1)} \langle \tilde{\omega}_i(\mathbf{k}) \tilde{\omega}_j^*(\mathbf{k}') \rangle L_i^A L_j^A \\
 &= - \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \int d^3k \int d^3k' e^{i\mathbf{k} \cdot (-\hat{p}_A(t-t_2)) - i\mathbf{k}' \cdot (-\hat{p}_A(t-t_1))} \langle \tilde{\omega}_i(\mathbf{k}) \tilde{\omega}_j^*(\mathbf{k}') \rangle L_i^A L_j^A \\
 &= - \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \int dk k^2 \frac{g(k)}{k^2} \int d\hat{k} e^{i\mathbf{k} \cdot \hat{p}_A(t_2-t_1)} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) L_i^A L_j^A
 \end{aligned}$$

where we have used that

$$\delta \mathcal{L}_t = -\omega(\mathbf{r}(t)) \cdot \mathbf{L}, \quad \omega(\mathbf{r}) = \int d^3k e^{i\mathbf{k} \cdot \mathbf{r}} \tilde{\omega}(\mathbf{k}), \quad \mathbf{r}(t') = -\hat{p}(t-t'), \quad U^0 = 1.$$

and assumed for the turbulence tensor

$$\langle \tilde{\omega}_i(\mathbf{k}) \tilde{\omega}_j^*(\mathbf{k}') \rangle = \frac{g(k)}{k^2} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) \delta(\mathbf{k} - \mathbf{k}'), \quad g(k) = g_0 \left(\frac{k}{k_0} \right)^{-q}$$

Example: non-interacting diagram

Now, we substitute,

$$T \equiv t_2 - t_0 \quad \Rightarrow \quad dT = dt_2 \quad \text{and} \quad \tau \equiv t_2 - t_1 \quad \Rightarrow \quad d\tau = -dt_1$$

and, performing a plane wave expansion, we find

$$\begin{aligned} (\langle U_{t,t_0}^A U_{t,t_0}^{B*} \rangle)_{1a} &= - \int_0^{t-t_0} dT \int_0^T d\tau \int dk g(k) \sum_{\ell_A} (2\ell_A + 1) i^{\ell_A} j_{\ell_A}(k\tau) \int d\hat{k} P_{\ell_A}(\hat{k} \cdot \hat{p}_A) (\delta_{ij} - \hat{k}_i \hat{k}_j) \\ &= - \sum_{\ell_A} (2\ell_A + 1) i^{\ell_A} \underbrace{\left(\int_0^{t-t_0} dT \int_0^T d\tau \int dk g_0 j_{\ell_A}(k\tau) \right)}_{\equiv \Lambda_{\ell_A}(t-t_0)} \int d\hat{k} P_{\ell_A}(\hat{k} \cdot \hat{p}) (\delta_{ij} - \hat{k}_i \hat{k}_j) \end{aligned}$$

where by substituting $T = T' \Delta T$ and $\tau = \tau' T = \tau' T' \Delta T$,

$$\begin{aligned} \Lambda_{\ell_A}(\Delta T) &= \int_0^{\Delta T} dT \int_0^T d\tau \int dk g_0 j_{\ell_A}(k\tau) \\ &= g_0 k_0 (\Delta T)^2 \int_0^1 dT' T' \int_0^1 d\tau' \int_1^\infty dk' j_{\ell_A}(k_0 \Delta T k' \tau' T') \\ &= \gamma r^2 \int_0^1 dT' T' \int_0^1 d\tau' \int_1^\infty dk' j_{\ell_A}(r k' \tau' T') \end{aligned}$$

Here, we have defined $r \equiv k_0 \Delta T$ and $\gamma \equiv g_0/k_0$.

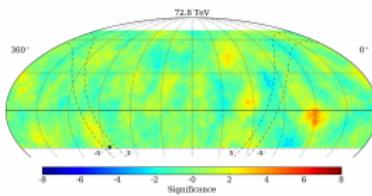
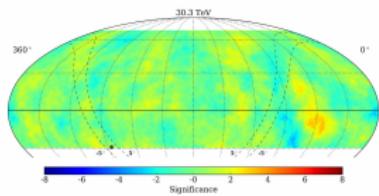
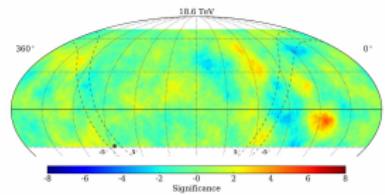
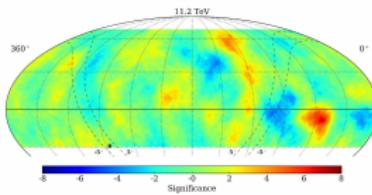
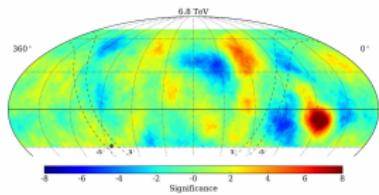
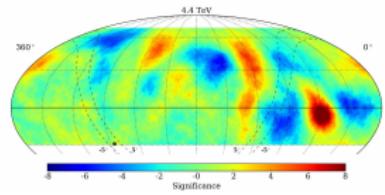
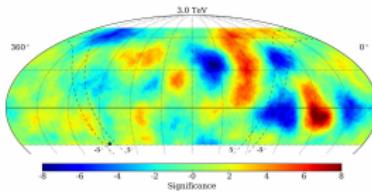
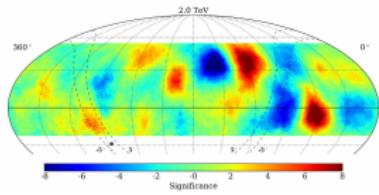
Backup

③ Diagrammatic technique

④ Observations

⑤ Quasi-linear theory

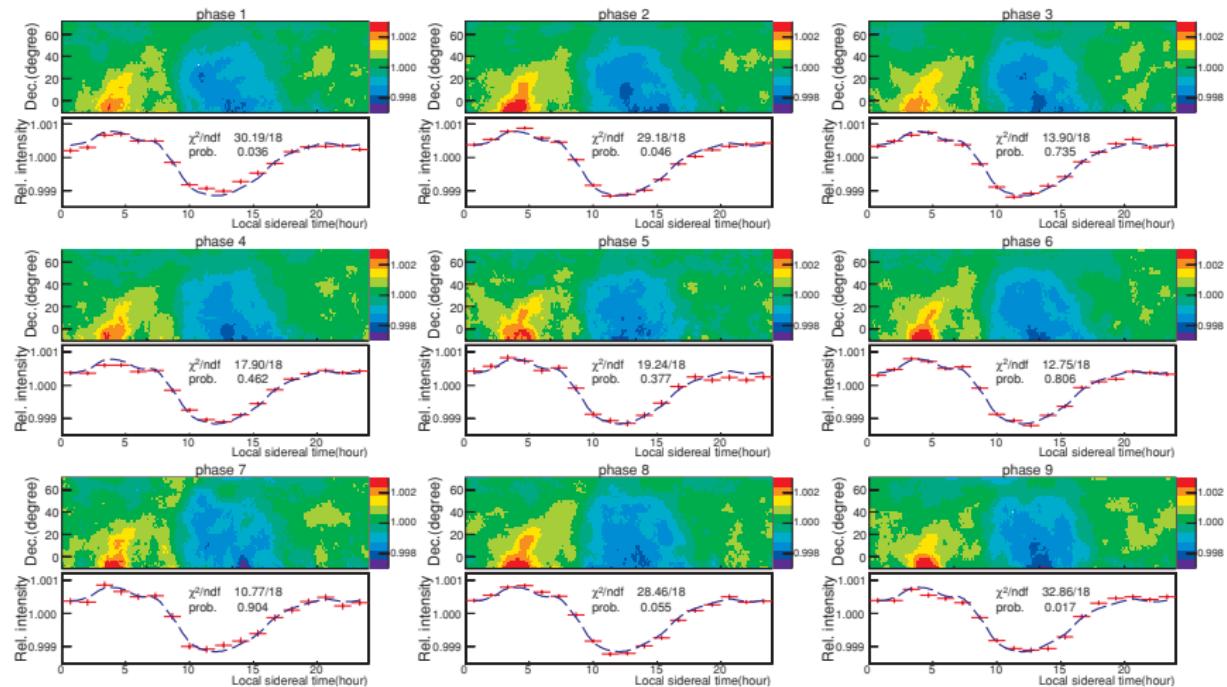
Energy dependence



Abeysekara et al., arXiv:1805.01847

Decrease of amplitude and flip of direction around 100 TeV also seen by IceCube

Time dependence



Amenomori et al., ApJ 711 (2010) 119

No significant time-dependence over 9 years.

Backup

③ Diagrammatic technique

④ Observations

⑤ Quasi-linear theory

Vlasov equation

- Liouville's theorem:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f = 0$$

- In a regular and turbulent magnetic field:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \delta \mathbf{B}(\mathbf{r}) \equiv p_0/e (\boldsymbol{\Omega} + \boldsymbol{\omega}(\mathbf{r}))$$

- Angular momentum operator $\mathbf{L} \equiv -i\mathbf{p} \times \nabla_{\mathbf{p}}$:

$$\dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f = \mathbf{p} \times (\boldsymbol{\Omega} + \boldsymbol{\omega}(\mathbf{r})) \cdot \nabla_{\mathbf{p}} f = -i(\boldsymbol{\Omega} + \boldsymbol{\omega}(\mathbf{r})) \cdot \mathbf{L} f$$

- Deterministic and stochastic operators \mathcal{L}_0 and $\delta \mathcal{L}$:

$$\frac{\partial f}{\partial t} + \underbrace{(\dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} - i\boldsymbol{\Omega} \cdot \mathbf{L}) f}_{\mathcal{L}_0} + \underbrace{(-i\boldsymbol{\omega} \cdot \mathbf{L}) f}_{\delta \mathcal{L}} = 0$$

Quasi-linear theory

e.g. Jokipii, Rev. Geophys. 9 (1971) 27

- Equations for averaged phase space density and fluctuations: $f = \langle f \rangle + \delta f$

$$\frac{\partial}{\partial t} \langle f \rangle + \mathcal{L}_0 \langle f \rangle = -\langle \delta \mathcal{L} \delta f \rangle,$$

$$\frac{\partial}{\partial t} \delta f + \mathcal{L}_0 \delta f \simeq -\delta \mathcal{L} \langle f \rangle.$$

- Integration along *unperturbed trajectories* $P(t')$

$$\delta f(t, \mathbf{r}, \mathbf{p}) \simeq \delta f(t_0, \mathbf{r}(t_0), \mathbf{p}(t_0)) - \int_{t_0}^t dt' \left[\delta \mathcal{L} \langle f \rangle \right]_{P(t')}$$

- Scattering term $\langle \delta \mathcal{L} \delta f \rangle$ can be approximated as

$$\langle \delta \mathcal{L} \delta f \rangle \simeq - \left\langle \delta \mathcal{L} \int_{-\infty}^t dt' \left[\delta \mathcal{L} \langle f \rangle \right]_{P(t')} \right\rangle \simeq \frac{\partial}{\partial \mu} D_{\mu\mu} \frac{\partial}{\partial \mu} \langle f \rangle$$

→ Pitch-angle diffusion → spatial diffusion